

Torsion, Chern-Simons Term and Diffeomorphism Invariance

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Abstract

In the *torsion* \otimes *curvature* approach of gravity Chern-Simons modification has been considered here. It has been found that Chern-Simons contribution to the bianchi identity has become cancelled from that of the scalar field part. But “homogeneity and isotropy” consideration of present day cosmology is a consequence of the “strong equivalence principle” and vice-versa.

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1 Introduction

Chern-Simons (CS) modified gravity is a 4-dimensional deformation of General Relativity (GR), postulated by Jackiw and Pi[1]. This theory of gravity is an extension of general relativity by adding a parity violating Pontryagin density $*RR$ coupled to a scalar field θ to Einstein-Hilbert Lagrangian. This scalar field θ can be viewed as either a prescribed background quantity or as an evolving dynamical field. The $*RR$ is defined as the contraction of the Riemann tensor with its dual and it is odd under a parity transformation, thus potentially enhancing gravitational parity-breaking. The CS correction introduces a means to enhance parity-violation through a pure curvature term, as opposed to through the matter sector, as more commonly

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happens in GR. One of the important feature of CS modified gravity is its emergence within predictive frameworks of more fundamental theories. For example, the low energy limit of string theory comprises general relativity with a parity violating correction term, that is nothing but the Pontryagin density. This term is crucial for cancelling gravitational anomaly in string theory through Green-Schwartz mechanism[2]. The Pontryagin density, as an anomaly cancelling term, also arises in particle physics and in the context of loop quantum gravity[3]. Ref.[3] gives a review on Chern-Simons Modified General Relativity. In a recent approach the CS modified gravity reduces to topologically massive gravity in three dimensions[4].

It is now a well established fact that de Sitter group is the correct underlying gauge group of gravity as unlike Poincare group, it is a semisimple group yielding consistent field equations. Thus de Sitter gauge theory comes up as the corrected Poincare gauge theory[5]. Recently a gravitational Lagrangian has been proposed[6], where a Lorentz invariant part of the de Sitter Pontryagin density has been treated as the Einstein-Hilbert Lagrangian. In this formalism the role of torsion in the underlying manifold is multiplicative rather than additive and the Lagrangian looks like $torsion \otimes curvature$. This indicates that torsion is uniformly nonzero everywhere. In the geometrical sense, this implies that micro local space-time is such that at every point there is a direction vector (vortex line) attached to it. This effectively corresponds to the non commutative geometry having the manifold $M_4 \times Z_2$, where the discrete space Z_2 is just not the two point space but appears as an attached direction vector[7]. In this approach we consider only a particular class of U_4 space, where only the axial vector part of the torsion is present everywhere in space time.¹ This may be compared with another approach[10] where the axial vector torsion is given by the derivative of a pseudoscalar field and then one gets a propagating torsion wave unlike in the standard Einstein-Cartan theory where torsion ceases to exist outside spinorial matter. This propagating torsion extends over whole spacetime. Some authors have pointed that only scalar or pseudoscalar torsion modes can be used to investigate the torsion cosmology[11, 12]. Their conclusion emerges due to consideration of additive torsion terms in the gravitational part of the Lagrangian. In particular the topological Nieh-Yan term, which is a pseudoscalar density, is the term from the torsion sector which plays an important role in the present formalism. Here the additive torsion decouples from the theory but not the multiplicative one and this implies the non trivial omnipresence of the axial torsion. In this model of minimum extension of Einstein-Hilbert

¹It is to be noted here that in the presence of spinorial matter only the axial vector part of the torsion couples to the spinor field[8, 9].

theory, we only take axial torsion as an extra degree of freedom. Considering torsion and torsion-less connection as independent fields[13], it has been found that, κ of Einstein-Hilbert Lagrangian appears as an integration constant and is linked with the topological Nieh-Yan density of U_4 space. As a result, κ has got its definite geometrical meaning in U_4 space in comparison to its standard meaning of being simply an ad hoc constant. If we consider axial vector torsion together with a scalar field ϕ connected to a local scale factor[14], then the Euler-Lagrange equations also link, in laboratory scale, the mass of the scalar field with the Nieh-Yan density and, in cosmic scale of FRW-cosmology, they predict only three kinds of the phenomenological energy density representing mass, radiation and cosmological constant. In a recent paper[15], it has been shown that this scalar field may also be interpreted to be linked with the dark matter and dark radiation. Recently it has been shown that[16], using field equations of all fields except the frame field, the starting Lagrangian reduces to a generic $f(\mathcal{R})$ gravity Lagrangian which, for FRW metric, gives standard FRW cosmology. But for non-FRW metric, in particular of Ref.[17], with some particular choice of the functions of the scalar field ϕ one gets $f(\mathcal{R}) = f_0 \mathcal{R}^{1+v_{tg}^2}$, where v_{tg} is the constant tangential velocity of the stars and gas clouds in circular orbits in the outskirts of spiral galaxies. In this letter we are going to study the CS modification of this formalism where the CS scalar θ has been considered to be a function of the scalar field ϕ .

2 Formulation

The gravitational Lagrangian, with CS modification, may be defined to be[3, 14, 16]

$$\begin{aligned} \mathcal{L}_G + \mathcal{L}_{CS} = & N\{\mathcal{R} - u(\phi)\} + {}^*BB \\ & \mp \frac{1}{2}d\phi \wedge {}^*d\phi - h(\phi)\eta + \frac{1}{4}\theta(\phi)\hat{R}^{ab} \wedge \hat{R}_{ab}, \end{aligned} \quad (1)$$

where $*$ is Hodge duality operator, $\mathcal{R}\eta = \frac{1}{2}\bar{R}^{ab} \wedge \eta_{ab}$, $B = B_a \wedge \bar{\nabla}e^a$, $\bar{R}^b{}_a = d\bar{\omega}^b{}_a + \bar{\omega}^b{}_c \wedge \bar{\omega}^c{}_a$, $\hat{R}^b{}_a = d\hat{\omega}^b{}_a + \hat{\omega}^b{}_c \wedge \hat{\omega}^c{}_a$, $\bar{\omega}^a{}_b = \omega^a{}_b - T^a{}_b$, $\hat{\omega}^b{}_a = -e^{b\nu}e_{a\nu;\mu}dx^\mu$, $T^a = \frac{1}{2!}e^{a\mu}T_{\mu\nu\alpha}dx^\nu \wedge dx^\alpha$, $T^{ab} = e^{a\mu}e^{b\nu}T_{\mu\nu\alpha}dx^\alpha$, $T = \frac{1}{3!}T_{\mu\nu\alpha}dx^\mu \wedge dx^\nu \wedge dx^\alpha$, $N = dT$, $\eta_a = \frac{1}{3!}\epsilon_{abcd}e^b \wedge e^c \wedge e^d$ and $\eta_{ab} = {}^*(e_a \wedge e_b)$. Here $\bar{\nabla}$ represents exterior covariant differentiation with respect to the connection one form $\bar{\omega}^{ab}$, $:$ represents tensorial covariant differentiation, w.r.t. the Christoffel connection, acting upon external indices and B_a is a two form with one internal index and of dimension $(length)^{-1}$ and $u(\phi)$, $h(\phi)$, $\theta(\phi)$ are unknown

functions of ϕ whose forms are to be determined subject to the geometric structure of the manifold.

Now we write the total gravity Lagrangian in the presence of a spinorial matter field, given by

$$\mathcal{L}_{tot.} = \mathcal{L}_G + \mathcal{L}_{CS} + \mathcal{L}_D, \quad (2)$$

where[15]

$$\begin{aligned} \mathcal{L}_D = & \left[\frac{i}{2} \{ \bar{\psi}^* \gamma \wedge D\psi + \overline{D\psi} \wedge * \gamma \psi \} - \frac{g}{4} \bar{\psi} \gamma_5 \gamma \psi \wedge T \right. \\ & \left. + c_\psi \sqrt{*dT} \bar{\psi} \psi \eta \right] \end{aligned} \quad (3)$$

$$\gamma_\mu := \gamma_a e^a_\mu, \quad * \gamma := \gamma^a \eta_a, \quad D := d + \Gamma \quad (4)$$

$$\begin{aligned} \Gamma &:= \frac{1}{4} \gamma^\mu D^{\{\} } \gamma_\mu = \frac{1}{4} \gamma^\mu \gamma_{\mu;\nu} dx^\nu \\ &= -\frac{i}{4} \sigma_{ab} e^{a\mu} e^b_{\mu;\nu} dx^\nu \end{aligned} \quad (5)$$

here $D^{\{\} }$, or $:$ in tensorial notation, is Riemannian torsion free covariant differentiation acting on external indices only; $\sigma^{ab} = \frac{i}{2}(\gamma^a \gamma^b - \gamma^b \gamma^a)$, $\bar{\psi} = \psi^\dagger \gamma^0$ and g, c_ψ are both dimensionless coupling constants. Here ψ and $\bar{\psi}$ have dimension $(length)^{-\frac{3}{2}}$ and conformal weight $-\frac{1}{2}$. It can be verified that under $SL(2, C)$ transformation on the spinor field and gamma matrices, given by,

$$\begin{aligned} \psi \rightarrow \psi' &= S\psi, \quad \bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi} S^{-1} \\ \text{and } \gamma \rightarrow \gamma' &= S\gamma S^{-1}, \end{aligned} \quad (6)$$

where $S = \exp(\frac{i}{4}\theta_{ab}\sigma^{ab})$, Γ obeys the transformation property of a $SL(2, C)$ gauge connection, i.e.

$$\Gamma \rightarrow \Gamma' = S(d + \Gamma)S^{-1} \quad (7)$$

$$\text{s. t. } D\gamma := d\gamma + [\Gamma, \gamma] = 0. \quad (8)$$

As in Ref.[14, 15], by varying the independent fields in the Lagrangian $\mathcal{L}_{tot.}$, we obtain the Euler-Lagrange equations and then after some simplification we get the following results

$$\bar{\nabla} e_a = 0, \quad (9)$$

$$*N = \frac{1}{\kappa}, \quad (10)$$

i.e. $\bar{\nabla}$ is torsion free and κ is an integration constant having dimension of

$(length)^2$.²

$$m_\psi = c_\psi \sqrt{*dT} = \frac{c_\psi}{\sqrt{\kappa}}, \quad (11)$$

$$\begin{aligned} i^* \gamma \wedge D\psi - \frac{g}{4} \gamma_5 \gamma \wedge T\psi + m_\psi \psi \eta &= 0, \\ i \overline{D\psi} \wedge * \gamma - \frac{g}{4} \overline{\psi} \gamma_5 \gamma \wedge T + m_\psi \overline{\psi} \eta &= 0. \end{aligned} \quad (12)$$

$$(G^b_a + C^b_a) \eta = -\kappa \left[\frac{i}{8} \{ \overline{\psi} (\gamma^b D_a + \gamma_a D^b) \psi - (\overline{D_a \psi} \gamma^b + \right. \quad (13)$$

$$\left. \overline{D^b \psi} \gamma_a) \psi \} \eta - \frac{g}{16} \overline{\psi} \gamma_5 (\gamma_a^* T^b + \gamma^{b*} T_a) \psi \eta \right. \quad (14)$$

$$\left. \pm \frac{1}{2} \partial_a \phi \partial^b \phi \eta + \frac{1}{2} h \eta \delta^b_a \right] \quad (15)$$

$$= -\kappa [T^b_a(\psi) + T^b_a(\phi)] \eta \text{ say,} \quad (16)$$

$$\begin{aligned} 0 &= \left[\frac{1}{2} \overline{\nabla}_\nu \overline{\psi} \left\{ \frac{\sigma^b_a}{2}, \gamma^\nu \right\} \psi + \frac{i}{2} \{ \overline{\psi} (\gamma^b D_a - \gamma_a D^b) \psi \right. \\ &\quad \left. - (\overline{D_a \psi} \gamma^b - \overline{D^b \psi} \gamma_a) \psi \} \right. \end{aligned} \quad (17)$$

$$\left. - \frac{g}{4} \overline{\psi} \gamma_5 (\gamma_a^* T^b - \gamma^{b*} T_a) \psi \right] \eta, \quad (18)$$

$$C^{ab} = \kappa [\overline{\nabla}_c \theta \epsilon^{cde(a} \overline{\nabla}_e \mathcal{R}^b)_{d} + \overline{\nabla}_c \overline{\nabla}_d \theta^* \bar{R}^{d(ab)c}], \quad (19)$$

$$T^b_a(\psi) = \frac{i}{8} \{ \overline{\psi} (\gamma^b D_a + \gamma_a D^b) \psi - (\overline{D_a \psi} \gamma^b + \quad (20)$$

$$\left. \overline{D^b \psi} \gamma_a) \psi \} - \frac{g}{16} \overline{\psi} \gamma_5 (\gamma_a^* T^b + \gamma^{b*} T_a) \psi, \quad (21)$$

$$T^b_a(\phi) = \pm \frac{1}{2} \partial_a \phi \partial^b \phi + \frac{1}{2} h \delta^b_a, \quad (22)$$

$$\kappa d \left[\frac{g}{4} * (\overline{\psi} \gamma_5 \gamma \psi \wedge T) + \Sigma \right] = -\frac{g}{4} \overline{\psi} \gamma_5 \gamma \psi, \quad (23)$$

$$\Sigma = \mp \frac{1}{2} * (d\phi \wedge * d\phi) + 2h - \frac{1}{\kappa} u \quad (24)$$

$$\left\{ \frac{1}{\kappa} u'(\phi) - h'(\phi) \right\} \eta + \frac{1}{4} \theta'(\phi) \hat{R}^{ab} \wedge \hat{R}_{ab} = \mp d^* d\phi. \quad (25)$$

²In (1), $\overline{\nabla}$ represents a $SO(3,1)$ covariant derivative, it is only on-shell torsion-free through the field equation (9). The $SL(2, C)$ covariant derivative represented by the operator D is torsion-free by definition, i.e. it is torsion-free both on on-shell and off-shell. Simultaneous and independent use of both $\overline{\nabla}$ and D in the Lagrangian density (2) has been found to be advantageous in the approach of this article. This amounts to the emergence of the gravitational constant κ to be only an on-shell constant and this justifies the need for the introduction of the Lagrangian multiplier B_a which appears twice in the Lagrangian density (1) such that $\bar{\omega}^a_b$ and e^a become independent fields.

Now we see that[1, 3]

$$\bar{\nabla}_b C^b{}_a = -\frac{1}{8}\kappa\partial_a\theta^*RR \quad (26)$$

$$\bar{\nabla}_b T^b{}_a(\phi) = \bar{\nabla}_b[\pm\frac{1}{2}\partial_a\phi\partial^b\phi + \frac{1}{2}h\delta^b{}_a] = \frac{1}{8}\partial_a\theta^*RR + \frac{1}{2}\partial_a\Sigma \quad (27)$$

Therefore Bianchi identity of $G^b{}_a$ implies

$$\bar{\nabla}_b T^b{}_a(\psi) = -\frac{1}{2}\partial_a\Sigma \quad (28)$$

Earlier[14, 15, 18], we have seen that $\Sigma = \text{constant}$ gives us standard isotropic and homogeneous FRW-universe at cosmic scale and from the last equation we see that this is the case of strong equivalence principle. Therefore we can state that **Isotropy & Homogeneity** \Leftrightarrow **Diffeomorphism Invariance**. Moreover, for $\Sigma = \text{constant}$, we can define conserved axial current[18] by

$$J \equiv \kappa^*(\bar{\psi}\gamma_5\gamma\psi \wedge T)T \quad (29)$$

The general theory of relativity is a diffeomorphism invariant theory where energy momentum tensor is covariantly conserved and there exists a killing field that generates an isometry of the spacetime. Now the vital question is which solution of Einstein's equation corresponds to our universe or at least an idealized model that approximates our universe. We know that the structure of the universe as predicted by GR based on simple cosmological principle of homogeneity and isotropy approximation in the large scale structure of our universe. So diffeomorphism invariance of Einstein's equation and simple cosmological principle are closely related. In our present formalism of minimal extension of GR we see that diffeomorphism invariance is still maintained if and only if the universe is isotropic and homogeneous. The equation (28) is valid with or without CS term of the action. So this conclusion remains valid even in the presence of CS modified extended GR. When we consider other form of cosmic energy density, may be in the early universe, we have to adopt a non-FRW geometry where we may have to forgo the isotropy and homogeneity of the universe[15], then the above result shows that conservation of energy-momentum tensor of baryonic matter is violated but the total energy-momentum tensor of both baryonic and dark matter/radiation is always conserved.

3 Discussion

In the standard formulation of Chern-Simons modified gravity where θ is taken as external variable, the presence of Cotton tensor $C^b{}_a$ in the modified

Einstein's equation violates the diffeomorphism invariance because the non vanishing divergence of cotton tensor is proportional to $*RR$, the Pontryagin density. To maintain the diffeomorphism invariance the consistency of dynamics forces $*RR$ to vanish, so the consistency condition suppressed the symmetry breaking CS term in the action, even though its variation results in the modified equation of motion[19]. When θ is taken as local dynamical variable then this constraint is replaced by evolution equation of θ which can be viewed as Klein-Gordon equation in the presence of a potential and Pontryagin density as source term[3]. Then from modified Einstein's equation it can be shown that strong equivalence principle is satisfied provided θ satisfies its evolution equation. So there is no need to impose Pontryagin constraint to maintain Lorentz symmetry and thereby strong equivalence principle.

In the first order formalism of CS modified gravity[10], curvature tensor is taken as $SO(3,1)$ field strength such that the gravitational part gives the standard first order Einstein-Hilbert action but the CS part $*RR$, being not torsion free, serves as the effect of the CS deformation on the space-time geometry through an effective contribution of torsion which depends on the external field θ [10]. In the absence of CS term the action reduces to standard Einstein-Hilbert action. But in our model of minimal extension of Einstein-Hilbert action a multiplicative torsion is already present in the lagrangian which looks like *torsion* \otimes *curvature*. The additive torsion may decouples from the theory but not the multiplicative one. This indicates that torsion is uniformly nonzero everywhere. This torsion is not present in the curvature tensor and in the CS term. Here we take the CS term as $\theta(\phi)\hat{R}^{ab}\wedge\hat{R}_{ab}$, where \hat{R}^{ab} is the curvature 2-form from the metric only.

In our formulation of minimal extension of GR the evolution equation for ϕ is given by equation (25). Here we see that strong equivalence principle is not automatically satisfied but it depends upon the isotropy and homogeneity of spacetime (28). This is because here we take CS modification of minimal extended gravity, ϕ is taken as dark matter field and θ is taken as a function of ϕ . As a consequence we see that energy momentum conservation of ordinary (baryonic) matter depends upon the constancy of Σ (or isotropy and homogeneity of spacetime). And therefore strong equivalence principle of ordinary baryonic matter depends entirely upon the evolution of dark matter to form the FRW-geometry of the present universe in cosmic scale.

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